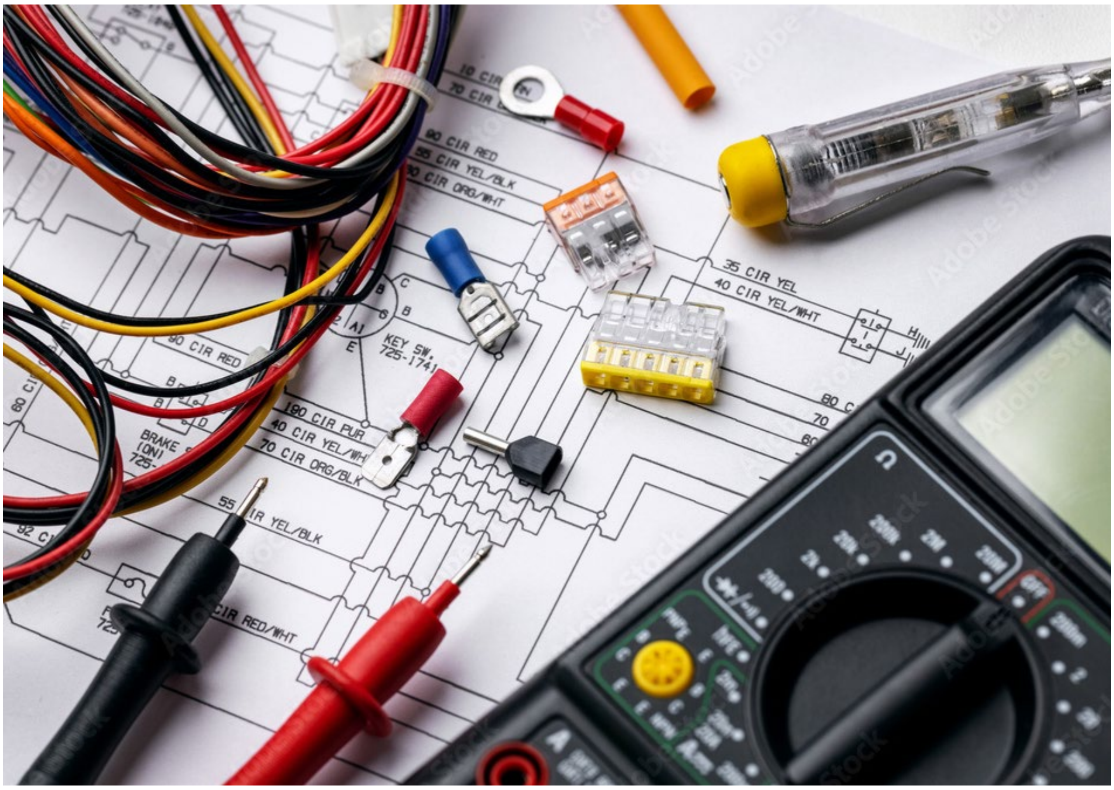

Circuit & Field Theory
Circuit Theory & Control

LAPLACE TRANSFORM

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Laplace Transform

The one sided Laplace transform of causal signals f(t) = 0 for t < 0, is defined as

$$F(s) = L[f(t)] = \int_0^\infty f(t)e^{-st}dt \qquad (1)^1$$

where s = σ + jω (complex frequency variable)

L = Laplace transform notation

The inverse Laplace transform of f(t) of F(s) and is given by the following complex domain integral

$$f(t) = \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} F(s)e^{-st}ds \qquad (2)$$

Symbolically transform pair(eq.1 and 2) may be written as

$$F(s) = L[f(t)] \qquad (3a)$$

$$f(t) = L^{-1}[F(s)] \qquad (3b)$$

Also $$f(t) \leftrightarrow F(s) \qquad (4)$$

LAPLACE TRANSFORM VS. FOURIER TRANSFORM

- The jω or ω in the Fourier transform has the same position s in the Laplace transform.
- The limits of integration in the two transforms are different. In the Laplace transform it is one sided i.e. from 0 to ∞ whereas in Fourier transform it is two sided i.e. from -∞ to ∞.
- The contours of integrations in the inverse transforms are different. In the Fourier transform it is along the imaginary axis whereas in the Laplace transform it is displaced by σ.
- The Laplace transform of f(t) is identical with the Fourier transform of f(t) multiplied by the convergence factor $e^{-\sigma t}$.

Example

Find the Laplace transform of

i. *δ(t), an impulse function*

ii. *u(t), a unit step function*

iii. $e^{-\alpha t}u(t)$

[1] In some books X(s) or I(s) is used instead of F(s).

Solution

(i) As per definition

$$L[\delta(t)] = \int_0^\infty \delta(t)e^{-st}dt$$
$$= \int_0^\infty \delta(t)\left(e^{-st}\Big|_{st=0}\right)dt$$
$$= \int_0^\infty \delta(t)dt$$
$$= 1$$

or $\delta(t) \leftrightarrow 1$ (5)

(ii)

$$L[u(t)] = \int_0^\infty u(t)e^{-st}dt$$
$$= \int_0^\infty e^{-st}dt$$
$$= -\frac{1}{s}e^{-st}\Big|_0^\infty$$
$$= 1/s$$

$u(t) \leftrightarrow 1/s$ (6)

(iii)

$$L[e^{-\alpha t}u(t) = \int_0^\infty e^{-\alpha t}e^{-st}dt$$
$$= \int_0^\infty e^{-(\alpha+s)}dt$$
$$= \frac{1}{s+\alpha}$$

$e^{-\alpha t}u(t) \leftrightarrow \frac{1}{s+\alpha}$ (7)

Example

Find Laplace transform of $\int_0^t f(t)dt$ *, given that L[f(t)] = F(s).*

Solution

$$L\left(\int_0^t f(t)dt\right) = \int_0^\infty \left(\int_0^t f(t)dt\right)e^{-st}dt$$

The integration is carried out by parts♠ where we let

♠ $\int u.dv = uv - \int v.du$

$$u = \int_0^t f(t)dt, du = f(t)dt$$

$$dv = e^{-st}dt, v = \frac{-1}{s}e^{-st}$$

Hence $$L\left(\int_0^t f(t)dt\right) = \frac{e^{-st}}{s}\int_0^t f(t)dt\Big|_0^\infty + \frac{1}{s}\int_0^\infty f(t)e^{-st}dt$$

Now, the first terms vanishes since e^{-st} approaches zero for infinite t and at lower limit $\int_0^t f(t)dt\Big|_{t=0} = 0$.

Hence $$L\left(\int_0^t f(t)dt\right) = -\frac{F(s)}{s}$$

TABLE OF LAPLACE TRANSFORM

Some of the transform pairs have already been obtained in the examples earlier. In table 1 some commonly used transform pairs are presented. You should memorize this table.

Table 1: Laplace Transform Pairs

f(t)	**F(s)**
$\delta(t)$	1
u(t)	1/s
t.u(t)	$1/s^2$
$t^n.u(t)$	$\frac{n!}{s^n + 1}$
$e^{-\alpha t}u(t)$	$1/(s+\alpha)$
$te^{-\alpha t}u(t)$	$1/(s+\alpha)^2$
$t^n e^{-\alpha t}u(t)$	$\frac{n!}{(s+\alpha)^{n+1}}$
$\cos\omega_0 t.u(t)$	$\frac{s}{s^2 + \omega_0^2}$
$\sin\omega_0 t.u(t)$	$\frac{\omega_0}{s^2 + \omega_0^2}$
$e^{-\alpha t}\cos\omega_0 t.u(t)$	$\frac{s+\alpha}{(s+\alpha)^2 + \omega_0^2}$
$e^{-\alpha t}\sin\omega_0 t.u(t)$	$\frac{\omega_0}{(s+\alpha)^2 + \omega_0^2}$

PROPERTIES OF LAPLACE TRANSFORM

Properties of the Laplace transform are helpful in obtaining Laplace transform of composite functions and in the solution of linear integro-differential equations. Since properties are proved below and other useful properties are presented in Table 2. memorize these properties of Laplace transform.

Table 2: Properties of Laplace Transform

Operation	**f(t)**	**F(s)**
Addition	$f_1(t) + f_2(t)$	$F_1(s) + F_2(s)$
Scalar multiplication	$\alpha f(t)$	$\alpha F(s)$
Time differentiation	df/dt or f'(t)	$sF(s) - f(0)$
	d^2f/dt^2 or f''(t)	$s^2F(s) - sf(0) - f'(0)$
Time integration	$\int_0^t f(t)dt$	$\frac{1}{s}F(s)$
	$\int_{-\infty}^t f(t)dt$	$\frac{1}{s}F(s) + \frac{1}{s}\int_{-\infty}^0 f(t)dt$
Time shift	$f(t - t_0)u(t - t_0)$	$F(s)e^{-st_0}$; $t_0 \geq 0$
Frequency Shift	$f(t)e^{\alpha t}$	$F(s - \alpha)$
Frequency differentiation	-tf(t)	dF(s)/ds
Frequency integration	f(t)/t	$\int_s^\infty F(s)ds$
Scaling	$f(\alpha t), \alpha \geq 0$	$\frac{1}{\alpha}F\left(\frac{s}{\alpha}\right)$
Time convolution	$f_1(t)*f_2(t)$	$F_1(s)F_2(s)$
Frequency convolution	$f_1(t)f_2(t)$	$F_1(s)*F_2(s)$
Initial value	f(0)	$\lim_{s\to\infty} sF(s)$
Final value	$f(\infty)$	$\lim_{s\to 0} sF(s)$

Linearity

$$f_1(t) \leftrightarrow F_1(s)$$

and $$f_2(t) \leftrightarrow F_2(s)$$

Then $$af_1(t) + bf_2(t) \leftrightarrow aF_1(s) + bF_2(s) \quad (8)$$

where a and b are constants.

Proof

$$L[af_1(t)+bf_2(t)] = a\int_0^\infty f_1(t)e^{-st}dt + b\int_0^\infty f_2(t)e^{-st}dt$$
$$= aF_1(s)+bF_2(s) \qquad (9)$$

Frequency Shift

If $f(t) \leftrightarrow F(s)$

Then $f(t)e^{\alpha t} \leftrightarrow F(s-\alpha)$ (10)

Proof

$$L[f(t)e^{\alpha t}] = \int_0^\infty f(t)e^{\alpha t}e^{-st}dt = \int_0^\infty f(t)e^{-(s-a)t}dt = F(s-\alpha) \qquad (11)$$

Time Shift

If $f(t) \leftrightarrow F(s)$

Then $f(t-t_0)u(t-t_0) \leftrightarrow F(s)e^{-st_0}$ (12)

Proof

$$L[f(t-t_0)u(t-t_0)] = \int_0^\infty f(t-t_0)u(t-t_0)e^{-st}dt = \int_0^\infty f(\lambda)u(\lambda)e^{-s(\lambda+t_0)}d\lambda \qquad (13)$$

where $(t - t_0) = \lambda$

As $f(\lambda)$ is causal, the lower limit in integral of Eq. 13 can be changed to 0. Thus

$$L[f(t-t_0)u(t-t_0)] = e^{-st_0}\int_0^\infty f(\lambda)e^{-s\lambda}d\lambda = e^{-st_0}F(s)$$

(14)

Time Differentiation

$$L\left(\frac{df(t)}{dt}\right) \leftrightarrow sF(s)-f(0) \qquad (15)$$

and, in general

$$L\left(\frac{d^n f(t)}{dt}\right) \leftrightarrow s^nF(s)-s^{n-1}f^1(0)-s^{n-2}f^1(0)...-f^{(n-1)}(0) \qquad (16)$$

Proof

$$L\left(\frac{df(t)}{dt}\right) = \int_0^\infty \frac{df(t)}{dt}e^{-st}dt$$

Integrating by parts, we get

$$L\left(\frac{df(t)}{dt}\right) = f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty f(t)e^{-st}dt \qquad (17)$$

Existence of F(s) guarantees $f(t)e^{-st}\Big|_{t=\infty} = 0$

Hence Eq. 17 becomes

$$L\left(\frac{df(t)}{dt}\right) = sF(s) - f(0) \qquad (18)$$

Time Integration

$$L\left(\int_0^t f(\tau)d\tau\right) \leftrightarrow \frac{F(s)}{s} \qquad (19)$$

and

$$L\left(\int_{-\infty}^t f(\tau)d\tau\right) \leftrightarrow \frac{F(s)}{s} + \frac{\int_{-\infty}^0 f(\tau)d\tau}{s} \qquad (20)$$

Initial Value Theorem

If the function f(t) and its derivative f'(t) are Laplace transformable then

$$f(0) = \lim_{s\to\infty} sF(s) \qquad (21)$$

Proof

We know that L[f'(t)] = sF(s) – f(0) [from eq. 18)

By taking the limit s → ∞ on both sides

$$\lim_{s\to\infty} L[f'(t)] = \lim_{s\to\infty}[sF(s) - f(0)]$$

or $$\lim_{s\to\infty}\int_0^\infty f'(t)e^{-st}dt = \lim_{s\to\infty}[sF(s) - f(0)]$$

As s→∞, the integration on L.H.S. becomes zero.

i.e. $$\int_0^\infty \lim_{s\to\infty}[f(t)e^{-st}]dt = 0$$

$$0 = \lim_{s\to\infty} sF(s) - f(0)$$

i.e. $$f(0) = \lim_{s\to\infty} sF(s)$$

Final Value Theorem

If f(t) and f'(t) are Laplace transformable then

$$f(\infty) = \lim_{s\to 0} sF(s) \qquad (22)$$

Proof

We know that L[f(t)] = sF(s) – f(0)

By taking the limit s →0 on both sides, we have

$$\lim_{s\to 0} L[f(t)] = \lim_{s\to 0}[sF(s) - f(0)]$$

or $$\lim_{s\to 0}\int_0^\infty f(t)e^{-st}dt = \lim_{s\to 0}[sF(s) - f(0)]$$

$$\int_0^\infty f(t)dt = \lim_{s\to 0}[sF(s) - f(0)]$$

But $$\left(f(t)\right)_0^\infty = \lim_{t\to\infty} f(t) - \lim_{t\to 0}$$

Hence $$\lim_{t\to\infty} f(t) - \lim_{t\to 0} = \lim_{s\to 0} sF(s) - f(0)$$

Since f(0) is not a function of S, it gets cancelled from both sides

Hence $$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

I.e. $$f(\infty) = \lim_{s\to 0} sF(s)$$

Time Scaling Theorem

A scale change is performed to a time variable function f(t) by introducing t_0 in the time domain where t_0 is a +ve constant. The new function being $f(t/t_0)$.

Now $$L[f(t/t_0)] = \int_0^\infty f(t/t_0)e^{-st}dt = t_0\int_0^\infty f(t/t_0)e^{-(t_0.s)t/t_0}d(t/t_0)$$

Let $$t/t_0 = T$$

$\therefore$ $$L[f(t/t_0)] = L[f(T)] = t_0\int_0^\infty f(T)e^{-t_0 sT}dT$$

$\therefore$ $$L[f(t/t_0)] = t_0F(t_0 s)$$

Complex Translation Theorem

Complex translation theorem states that

$$e^{-at}f(t) \leftrightarrow F(s+a)$$

Now $$F(s) = \int_0^\infty f(t)e^{-ab} = \int_0^\infty te^{-at}dt \text{ taking f(t) = 1}$$

on integration, we have

$$F(s) = \left[\frac{te^{-at}}{-a}\right]_0^\infty - \int_0^\infty e^{-at}dt = -\frac{1}{a}\left(te^{-at}\right)_0^\infty - \left(\frac{e^{-at}}{-a}\right)_0^\infty$$

$$= \frac{1}{a}(-1) = -\frac{1}{a}$$

$$= F(s+a)$$

Example

Find f(0) or f(0+) of the signal whose Laplace transform is

$$F(s) = \frac{(s+1)}{(s+3)(s+2)}$$

Solution

From initial value theorem

$$f(0) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{s(s+1)}{(s+3)(s+2)} = 1$$

Example

Find the final value of a continuous signal $x(t) = [2+e^{-3t}]u(t)$.

Solution

Laplace transform of signal x(t) can be found as

$$X(s) = L[x(t)] = L[(2+e^{-3t})u(t)]$$

or $$X(s) = \int_{-\infty}^{\infty} (2+e^{-3t})u(t)e^{-st}dt \qquad \text{(i)}$$

Using the definition of unit step function, we have

$$u(t) = \begin{cases} 1, t > 0 \\ 0, t < t \end{cases} \qquad \text{(ii)}$$

Substituting the value of u(t) in eq (i)

$$X(s) = \int_{0}^{\infty} (2+e^{-3t}).1.e^{-st}dt = \int_{0}^{\infty} [2e^{-st} + e^{-(s+3t)}]dt$$

or $$X(s) = 2\int_{0}^{\infty} e^{-st}dt + \int_{0}^{\infty} e^{-(s+3)t}dt = 2\left[\frac{1}{-s}e^{-st}\right] + \left[\frac{1}{-(s+3)}e^{-(s+3)t}\right]_0^{\infty}$$

or $$X(s) = \frac{2}{s} + \frac{1}{s+3} = \frac{2(s+3)+s}{s(s+3)} = \frac{3s+6}{s(s+3)} = \frac{3(s+2)}{s(s+3)} \qquad \text{(iii)}$$

Now $$x(\infty) = \lim_{t\to\infty}[sX(s)] \qquad \text{(iv)}$$

Substituting the value of X(s) in equation (iv), we get

$$x(\infty) = \lim_{s\to\infty}\left[s.\frac{3(s+2)}{s(s+3)}\right] = \lim_{s\to\infty}\left[\frac{3(s+2)}{(s+3)}\right] = \frac{3(0+2)}{(0+3)} = 2$$

Hence the final value of signal x(t) is 2.

Example

Find the initial value of the function whose Laplace transform is

$$V(s) = A\frac{(s+a)\sin\theta + b\cos\theta}{(s+a)^2 + b^2}$$

Solution

Applying initial value theorem we have

$$f(0) = \lim_{s\to\infty} sF(s)$$

$$= \lim_{s\to\infty} sA\frac{(s+a)\sin\theta + b\cos\theta}{(s+a)^2 + b^2}$$

$$= \lim_{s\to\infty} A\frac{s(s+a)\sin\theta + b\cos\theta}{(s+a)^2 + b^2}$$

Divide numerator and denominator by s^2.

$$f(0) = \lim_{s\to\infty} A\frac{\left(1+\frac{a}{s^2}\right)\sin\theta + \frac{b}{s^2}\cos\theta}{\left(1+\frac{a}{s}\right)^2 + \left(\frac{b}{s}\right)^2}$$

Applying the limit[2], f(0) = A sinθ.

Problem

Find the final value of the function whose Laplace transform is

$$F(s) = \frac{s+6}{s(s+3)}$$

Answer: 2

Problem

Find the initial value of the continuous signal if its Laplace transform is given as

$$X(s) = \frac{2s+1}{s^2-1}$$

Answer: 2

Problem

Find the initial and final values of function if its Laplace is given by

---

[2] For limit, read any standard mathematics book. In the present case, remember 1/∞ = 0.

$$X(s) = \frac{17s^3 + 7s^2 + s + 6}{s^5 + 3s^4 + 4s^2 + 2s}$$

Answer: 0, 3

Problem

Find the initial and final value of the current whose current transform I(s) is given by

$$I(s) = \frac{0.32}{s(s^2 + 2.42s + 0.672)}$$

Answer: 0, 0.477

Time Convolution

$$[f_1(t) * f_2(t)] \leftrightarrow F_1(s)_2(s) \qquad (23)$$

where

$$f_1(t) \leftrightarrow F_1(s)$$
$$f_2(t) \leftrightarrow F_2(s)$$

Proof

$$[f_1(t) * f_2(t)] = \int_0^t f_1(\lambda) f_2(t - \lambda) d\lambda \qquad \text{(i)}^3$$

As $f_1(t)$ and $f_2(t)$ are causal, upper limits in Eq. (i) can be changed from t to ∞. This is because $f_2(t) = 0$, $t<0$ or $f_2(t - \lambda) = 0$, $\lambda > t$. Thus

$$f_1(t) * f_2(t) = \int_0^\infty f_1(\lambda) f_2(t - \lambda) d\lambda \qquad \text{(ii)}$$

Taking the Laplace transform

$$L[f_1(t) * f_2(t)] = \int_0^\infty \left(\int_0^\infty f_1(\lambda) f_2(t - \lambda) \right) e^{-st} dt \qquad \text{(iii)}$$

Let $t - \lambda = \eta \rightarrow dt = d\eta$. By interchanging order of integrations, we can write Eq.(iii) as

$$L[f_1(t) * f_2(t)] = \int_0^\infty f_1(\lambda) \left(\int_0^\infty f_2(\eta) e^{-s\eta} d\eta \right) e^{-s\lambda} d\lambda \qquad \text{(iv)}$$

It then follows from Eq. (iv) that

$$L[f_1(t) * f_2(t)] = F_1(s) F_2(s) \qquad (24)$$

These and other properties of the Laplace transform are listed in table 2.

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[3] In some books τ is used instead of λ

Example

Evaluate the convolution integral when $x_1(t) = e^{-2t}$ *and* $x_2(t) = 2t$.

Solution

We know that the convolution of two integral functions $x_1(t)$ and $x_2(t)$ is expressed as

$$x_1(t) \otimes x_2(t) = \int_0^t x_1(\tau)x_2(t-\tau)$$

Then we have

$$x_1(t) \otimes x_2(t) = \int_0^t 2\tau e^{-2(t-\tau)} d\tau = e^{-2t}\int_0^t 2\tau e^{2\tau} d\tau$$

Simplifying, we get

$$x_1(t) \otimes x_2(t) = 2e^{-2t}\left[\tau\frac{e^{2\tau}}{2} - \int 1.\frac{e^{2\tau}}{2}\right]_0^t = 2e^{-2t}\left[\frac{te^{2\tau}}{2} - \frac{e^{2t}}{4} + \frac{1}{4}\right]$$

The last equation may be written as

$$x_1(t) \otimes x_2(t) = \left[t - \frac{1}{2} + \frac{e^{-2t}}{2}\right]u(t)$$

Example

Use the convolution theorem of Laplace transform to find $y(t) = x_1(t) \otimes x_2(t)$ *if* $x_1(t) = e^{-3t}u(t)$ *and* $x_2(t) = u(t-2)$.

Solution

We have $x_1(t) = e^{-3t}u(t)$

Therefore $X_1(s) = \dfrac{1}{s+3}$

Also $x_2(t) = u(t-2)$

Therefore $X_2(s) = \dfrac{e^{-2s}}{s}$

Hence $X(s) = X_1(s).X(s_2) = \dfrac{1}{s+3}.\dfrac{e^{-2s}}{s} = \dfrac{e^{-2s}}{s(s+3)}$

Problem

Determine convolution between two functions $f_1(t) = 2.u(t)$ *and* $f_2(t) = e^{-3t}.u(t)$ *where u(t) is a unit step function.*

Answer

$f_1(t)*f_2(t)=\frac{2}{3}[1-e^{-3t}]$

Hint: Use of Eq. 23 of time convolution

$f_1(t)*f_2(t) = F_1(s)F_2(s)$

$f_1(t) = 2u(t)$ $\quad\therefore F_1(s) = 2/s$ from table 1

$f_2(t) = e^{-3t}u(t)$ $\quad\therefore F_2(s) = 1/(s+3)$ from table 1

Example

Find the Laplace transform of cos ωt and Sin ωt.

Solution

As $e^{-\alpha t}u(t)\leftrightarrow\frac{1}{s+\alpha}$ (from table 1)

$$e^{-j\omega_0 t}u(t)\leftrightarrow\frac{1}{s+j\omega_0} \quad \text{(i)}$$

and

$$e^{+j\omega_0 t}u(t)\leftrightarrow\frac{1}{s-j\omega_0} \quad \text{(ii)}$$

Adding and subtracting Eq. (i) and (ii), we get

$$\cos\omega_0 t.u(t)\leftrightarrow\frac{s}{s^2+\omega_0^{\ 2}} \qquad \text{because } \cos\omega_0 t=\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}$$

and

$$Sin\omega_0 t.u(t)\leftrightarrow\frac{\omega_0}{s^2+\omega_0^{\ 2}} \qquad \text{because } Sin\omega_0 t=\frac{e^{j\omega_0 t}-e^{-j\omega_0 t}}{2}$$

Example

Find the Laplace transform of the following signal

$$f(t)=\frac{d}{dt}[e^{-\alpha t}u(t)] \quad \textit{(i)}$$

Solution

$$f(t) = -\alpha e^{-\alpha t}(t)u(t) + e^{-\alpha t}\delta(t) \quad \text{(ii)}$$

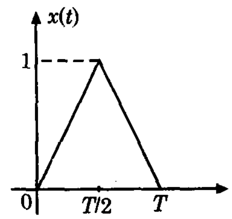
$$L[f(t)]=-\alpha\int_0^\infty e^{-\alpha t}e^{-st}dt+\int_0^\infty e^{-\alpha t}e^{-st}\delta(t)dt \quad \text{(iii)}$$

$$F(s)=\frac{-\alpha}{s+\alpha}+\int_0^\infty\delta(t)dt \quad \text{(iv)}$$

$$= \frac{-\alpha}{s+\alpha} + 1 = \frac{s}{s+\alpha} \quad \text{(v)}$$

Example

Find the Laplace transform of the triangular pulse.



Solution

According to figure, we have

$$x(t) = \begin{cases} \frac{2}{T}t & for\ 0 \le t \le \frac{T}{2} \\ 2 - \frac{2}{T}t & for\ \frac{T}{2} \le t \le T \end{cases} \quad \text{(i)}$$

We know that

$$X(s) = L[x(t)] = \int_0^\infty x(t)e^{-st}dt \quad \text{(ii)}$$

Substituting the value of x(t) from equation (i) into equation (ii), we obtain

$$X(s) = \int_0^{T/2}\left(\frac{2}{T}t\right)e^{-st}dt + \int_{T/2}^{T}\left(2 - \frac{2}{T}t\right)e^{-st}dt$$

or $$X(s) = \frac{2}{T}\int_0^{T/2} te^{-st}dt + 2\int_{T/2}^{T} e^{-st}dt - \frac{2}{T}\int_{T/2}^{T} t.e^{-st}dt$$

or $$X(s) = \frac{2}{T}\left[\left\{\frac{te^{-st}}{-s^2}\right\} - \left\{\frac{e^{-st}}{s^2}\right\}\right]_0^{T/2} + 2\left[\frac{e^{-st}}{-s}\right]_{T/2}^{T} - \frac{2}{T}\left[\left\{\frac{te^{-st}}{-s}\right\} - \left\{\frac{e^{-st}}{s^2}\right\}\right]_{T/2}^{T}$$

Simplifying, we get

$$X(s) = \frac{2}{T}\frac{1}{s^2} - \frac{4}{T}\frac{e^{-sT/2}}{s^2} + \frac{2}{T}.\frac{e^{-sT}}{s^2}$$

Example (AMIE Summer 2012, 6 marks)

Find Laplace transform of $\delta(t-T) + 3\delta(t) + \delta(t-3T)$

Solution

We know that

$$L\delta(t) = 1$$

and $$L\delta(t - t_1) = e^{-t_1 s}$$

Therefore given function becomes

$$L(t - T) = e^{-Ts}$$

$$L[3\delta(t)] = 3$$

and $$L(t - 3T) = e^{-3Ts}$$

Overall Laplace transform is

$$e^{-Ts} + 3 + e^{-3Ts}$$

Problem

Find the Laplace transform for the following signals:

(a) $\left(1 - e^{-3t}\right)u(t)$

(b) $\delta(t) - \delta(t-5)$

(c) $f(t) = \begin{cases} \sin t & 0 < t < \pi \\ 0 & oherwise \end{cases}$

(d) $\frac{1}{2a^3}(\sin at - at\cos at)u(t)$

Answer

(a) $F(s) = \left(\frac{1}{s} + \frac{1}{(3+s)}\right)$

(b) $F(s) = \left(1 - e^{5s}\right)$

(c) $F(s) = \frac{1}{(1+s^2)}\left(e^{-s\pi} + 1\right)$

(d) $F(s) = \frac{1}{(s^2 + a^2)^2}$

LAPLACE TRANSFORM OF PERIODIC FUNCTIONS

Time shift theorem is very useful in determining the transform of periodic time functions. let function x(t) be a casual periodic waveform which satisfies the condition x(t) = x(t + nT) for all t > 0 where T is the period of the function and n = 0, 1, 2, ……

Now, we can write

$$X(s) = \frac{1}{1 - e^{-sT}}\int_0^T x(t)e^{-st}dt = \frac{X_1(s)}{1 - e^{-sT}}$$

Example

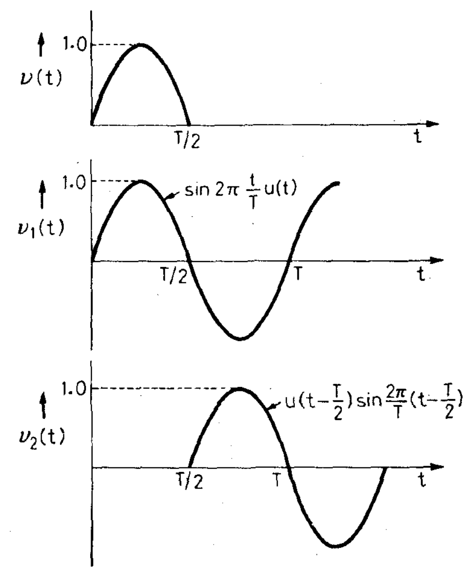
A half cycle sine wave function is given by $v(t) = \sin \omega t$. Determine Laplace transform.

Solution

$$v(t) = \sin \omega t = \sin \frac{2\pi t}{T} \qquad [\omega = 2\pi f = 2\pi / T]$$

This function will be positive for $0 \le t \le T/2$, T being the time period. Also it is observed that half cycle sine wave has *unity* amplitude.

Observing following figures, it is noticed that half cycle sine wave shown is actually a combination of two sine waves given by relations



$$v_1 = \left[\sin 2\pi \frac{t}{T}\right] u(t)$$

and $$v_2 = \left[\sin \frac{2\pi}{T}\left(t - \frac{T}{2}\right)\right] u\left(t - \frac{T}{2}\right)$$

when v_2 is shifted by T/2 from v_1.

Addition of v_1 and v_2 graphically gives the desired half cycle of the sine wave

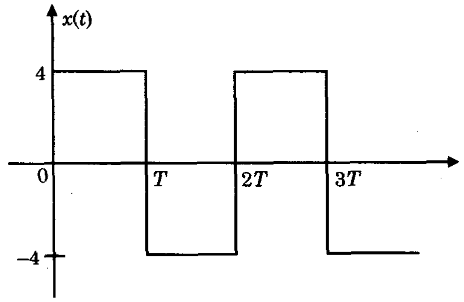
$$\therefore \qquad v(t) = \sin \frac{2\pi t}{T} u(t) + \sin \frac{2\pi}{T}\left(t - \frac{T}{2}\right) u\left(t - \frac{T}{2}\right)$$

Utilising Laplace transform methods

$$V(s) = \frac{2\pi / T}{s^2 + \left(\frac{2\pi}{T}\right)^2}\left[1 + e^{-\frac{T}{2}.s}\right]$$

Example

Determine the Laplace transform of the periodic rectangular waveform shown in figure.



Solution

Here the period is 2T.

$$L[X(t)] = \frac{1}{1-e^{-2sT}}\left[\int_0^{2T} x(t)e^{-st}dt\right]$$

or $$X(s) = \frac{1}{1-e^{-2sT}}\left[\int_0^{T} Ae^{-st}dt + \int_T^{2T}(-A)e^{-st}dt\right]$$

or $$X(s) = \frac{1}{1-e^{-2sT}}\left[\frac{-A}{s}(e^{-st})_0^T + \frac{A}{s}(e^{-st})_T^{2T}\right]$$

or $$X(s) = \frac{1}{1-e^{-2sT}}\left[\frac{A}{s}(e^{-sT}-1) + \frac{A}{s}(e^{-2sT}-e^{-sT})\right]$$

or $$\frac{1}{1-e^{-2sT}}\left[-\frac{A}{s}(e^{-sT}-1) + \frac{A}{s}(e^{-2sT}-e^{-sT})\right]$$

or $$X(s) = \frac{1}{1-e^{-2sT}}.\frac{A}{s}(1-2e^{-sT}+e^{-2sT})$$

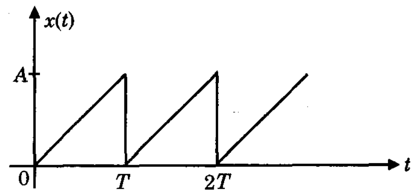
or $$X(s) = \frac{1}{1-e^{-2sT}}\left[\frac{A}{s}(1-e^{-sT})^2\right]$$

or $$X(s) = \frac{A}{s}\left[\frac{(1-e^{-sT})^2}{(1-e^{-sT})(1+e^{-sT})}\right]$$

or $$X(s) = \frac{A}{s}.\left[\frac{(1-e^{-sT})}{(1+e^{-sT})}\right] = \frac{A}{s}tahh\left(\frac{sT}{2}\right)$$

Problem

Obtain the Laplace transform of the given periodic waveform.



Answer: $X(s)=\dfrac{A}{Ts^2(1-e^{-sT})}[1-e^{-sT}-sTe^{-sT}]$

Hint: Period is T. For this

$$x(s)=\frac{1}{1-e^{-sT}}\left[\int_0^T x(t)e^{-st}dt\right]$$

INVERSE LAPLACE TRANSFORM

As per the definition of inverse Laplace transform

$$f(t)=\frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega}F(s)e^{st}ds \qquad (25)$$

Finding the inverse Laplace transform involves complex integration, which is cumbersome. However, using the uniqueness property of the Laplace transform, inverse transform can be found by looking at table 1. A rational F(s) can be first broken up into simple factors by partial fractioning. This procedure is demonstrated by some examples.

Example

Obtain the inverse Laplace transform of

(a) $F(s)=\dfrac{s^2+3s+1}{s(s+1)(s+2)}$

(b) $F(s)=\dfrac{1}{s^2(s+1)}$

(c) $F(s)=\dfrac{2s^2+6s+6}{(s+2)(s^2+2s+2)}$

Solution

(a)

Step 1: Let us first do partial fractioning of F(s)

$$F(s)=\frac{s^2+3s+1}{s(s+1)(s+2)}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}$$

[This is a case of distinct real poles]

$$=\frac{S(s+1)(s+2)+B(s)(s+2)+C(s)(s+1)}{s(s+1)(s+2)}$$

$$= \frac{A(s^2+2s+2)+B(s^2+2s)+C(s^2+s)}{s(s+1)(s+2)}$$

$$= \frac{(A+B+C)s^2+(3A+2B+C)+2A}{s(s+1)(s+2)}$$

Comparing coefficients of numerators on both sides

A + B + C = 1

3A +2B + C = 3

2A = 1

Solving A = 0.5, B = 1 and C = -0.5

$$F(s) = \frac{s^2+3s+1}{s(s+1)(s+2)} = \frac{0.5}{s} + \frac{1}{s+1} - \frac{0.5}{s+2}$$

Step 2: Using table (1), we get

$$f(t) = 0.5u(t) + e^{-t}u(t) - \frac{1}{2}e^{-2t}u(t)$$

(b)

Step 1: Let us first do partial fractioning of F(s)

$$F(s) = \frac{1}{s^2(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s+1)}$$ [This is a case of repeated poles]

$$= \frac{As(s+1)+B(s+1)+C(s^2)}{s^2(s+1)}$$

$$= \frac{A(s^2+s)+B(s+1)+Cs^2}{s^2(s+1)}$$

$$= \frac{(A+C)s^2+(A+B)s+B}{s^2(s+1)}$$

Comparing coefficients of numerators on both sides

B = 1

A + B = 0

A + C = 0

Solving

B = 1

A = -1

C = 1

$\therefore \qquad F(s) = \dfrac{1}{s^2(s+1)} = \dfrac{-1}{s} + \dfrac{1}{s^2} + \dfrac{1}{s+1}$

Step 2: From table (1), we get

$$f(t) = (t - 1 + e^{-t})u(t)$$

(c)

Step 1: Partial Fractioning

$$\frac{2s^2+6s+6}{(s+2)(s^2+2s+s)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+2s+2} = \frac{A(s^2+2s+2)+(Bs+C)(s+2)}{(s+2)(s^2+2s+2)}$$

Comparing coefficients of numerators on both sides

A = 1

B = 1

C = 2

Hence $F(s) = \dfrac{1}{s+2} + \dfrac{s+2}{s^2+2s+2} = \dfrac{1}{s+2} + \dfrac{s+1}{(s+1)^2+1} + \dfrac{1}{(s+1)^2+1}$

Step 2: From table (1), We get

$$f(t) = \left(e^{-2t} + e^{-t}\cos t + e^{-t}\sin t\right)u(t)$$

Example

Find inverse Laplace transform of

$$\frac{3s^2+s+1}{2s^2+3s}$$

Solution

Here powers of numerator and denominator are same. Hence first divide numerator by denominator. We get

$$F(s) = 1 + \frac{1-2s}{2s\left(s+\frac{3}{2}\right)}$$

Now let us first partial fraction $\dfrac{1-2s}{2s(s+3/2)}$ by using method already discussed in above examples.

We find $\quad \dfrac{1-2s}{2s(s+3/2)} = \dfrac{1}{3s} - \dfrac{4}{3(s+3/2)}$

$$\therefore \quad F(s) = 1 + \frac{1}{3s} - \frac{4}{3(s+3/2)}$$

$\therefore$ Inverse of F(s) = $\delta(t) + \frac{1}{3}u(t) - \frac{4}{3}e^{-(3/2)t}$

Problem

Find the inverse Laplace transform of

(a) $\frac{1}{(s+1)^2}$

(b) $\frac{s+4}{s^2+10s+24}$

(c) $\frac{2s+3}{(s+1)(s^2+4s+5)}$

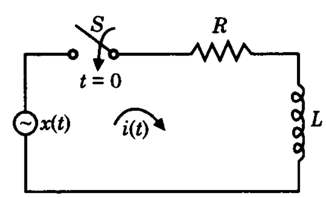
Answer

(a) $te^{-2t}u(t)$ (b) $e^{-6t}\,u(t)$ (c) $0.5[(3\sin t - \cos t)e^{-2t} + e^{-t}]u(t)$

APPLICATIONS OF LAPLACE TRANSFORMATION IN NETWORK ANALYSIS

Step Response of Series R-L Circuit

In the series RL circuit shown in figure, let us consider that the switch S is closed at time t = 0.



For the step response, the input excitation is x(t) = V_0.u(t). Applying Kirchoff's voltage law to the circuit, we get following diff equation:

$$L\frac{di(t)}{dt} + Ri(t) = V_0.u(t)$$

Taking Laplace transform, the last equation becomes

$$L[sI(s) - i(0+)] + RI(s) = \frac{V_0}{s}$$

Because of presence of inductance L, i(0+) = 0, i.e. the current through an inductor can not change instantaneously due to conservation of flux linkages.

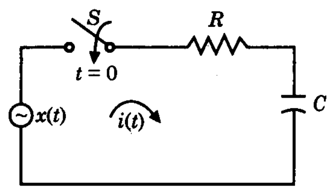
Therefore $$I(s) = \frac{V_0}{L}.\frac{1}{s\left(s-\frac{R}{L}\right)} = \frac{V_0}{L}.\frac{L}{R}\left[\frac{1}{s} - \frac{1}{s+(R/L)}\right] = \frac{V_0}{R}\left[\frac{1}{s} - \frac{1}{s+R/L}\right]$$

Taking inverse Laplace transform

$$i(t) = \frac{V_0}{R}\left[1 - e^{R/L}\right]$$

Step Response of Series R-C Circuit

For the given circuit, integral-differential eq. is



$$\frac{1}{C}\int_{-\infty}^{t} i(t)dt - Ri(t) = V_0 u(t)$$

This can be written as

$$\frac{1}{C}\int_{0}^{t} i(t)dt + \frac{1}{C}\int_{-\infty}^{0} i(t)dt + Ri(t) = V_0 u(t)$$

Taking Laplace transform, the last equation becomes

$$\frac{1}{C}\left[\frac{I(s)}{s}\right] + \frac{1}{C}L[q(0^+)] + RI(s) = \frac{V_0}{s}$$

or $$\frac{1}{C}\left[\frac{I(s)}{s} + \frac{q(0^+)}{s}\right] + RI(s) = \frac{V_0}{s}$$

Now $q(0^+)$ is the charge on the capacitor C at time t = 0^+. If the capacitor is initially uncharged, then $q(0^+) = 0$.

Hence $$I(s)\left[\frac{1}{Cs} + R\right] = \frac{V_0}{s}$$

or $$I(s) = \frac{V_0/R}{s + \frac{1}{RC}}$$

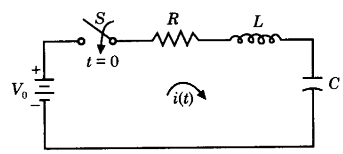
Therefore $$i(t) = \frac{V_0}{R}e^{t/RC}$$

Step Response of Series RLC Circuit

Following figure shows the series RLC circuit.

Integral Diff equation is

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_{-\infty}^{t} i(t)dt = V_0 u(t) \qquad \text{(i)}$$



Eq. (i) can be written as

$$L\frac{di(t)}{dt} Ri(t) + \frac{1}{C}\int_{-\infty}^{t} i(t)dt + \frac{1}{C}\int_{0}^{t} i(t)dt$$

Taking Laplace transform, the last equation would become

$$L(sI(s) - i(0^+)] + RI(s) + \frac{1}{C}L[q(0^+)] + \frac{1}{C}.\frac{I(s)}{s} = \frac{V_0}{s}$$

or $$L[sI(s) - i(0^+)] + RI(s) + \frac{1}{C}\frac{q(0^+)}{s} + \frac{1}{C}.\frac{I(s)}{s} = \frac{V_0}{s}$$

Now due to the presence of indicator L, we have i(0^+). Also, q(0^+) is the charge on the capacitor C at t = 0^+. If the capacitor is initially uncharged, then q(0^+) = 0. Putting these two initial conditions in last equation, we get

$$LsI(s) + RI(s) + \frac{I(s)}{C_s} = \frac{V_0}{s}$$

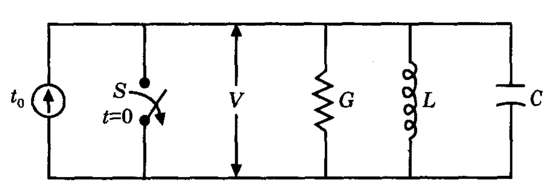
or $$I(s)\left[Ls + R + \frac{1}{Cs}\right] = \frac{V_0}{s} = \frac{V_0}{Ls^2 + Rs + \frac{1}{C}} = \frac{V_0}{l(s-p_1)(s-p_2)}$$

where $$p_1, p_2 = \frac{-R}{2L} \pm \frac{1}{2L}\sqrt{R^2 - 4\frac{L}{C}}$$

Hence $$i(t) = \frac{V_0 / L}{(p_1 - p_2)}[e^{p_1 t} - e^{p_2 t}]$$

Step Response of Series RLC Circuit

Following figure shows the circuit of parallel RLC circuit.



Let the switch S be opened at time t = 0, thus connecting the d.c. current source I_0 to the circuit.

Applying Kirchoff's current law to the circuit, we get the following integro-differential equation

$$C\frac{dV}{dt} + GV + \frac{1}{L}\int_{-\infty}^{t} Vdt = I_0 u(t) \qquad \text{(i)}$$

The last equation can be written as under

$$C\frac{dV}{dt} + GV + \frac{1}{L}\int_{-\infty}^{0} Vdt + \frac{1}{L}\int_{0}^{t} Vdt = I_0 u(t)$$

Taking Laplace transform, the last equation becomes

$$C[sV(s) - V(0^+) + GV(s) + \frac{1}{L}[\phi(0^+)] + \frac{1}{L}.\frac{V(s)}{s} = \frac{I_0}{s}$$

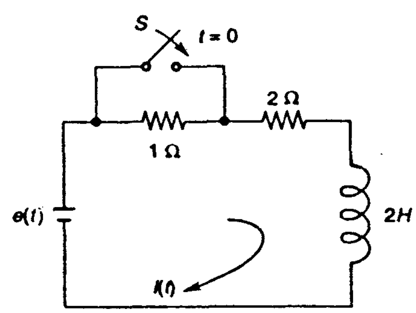
where $\phi(0^+)$ is the flux linkage and equals to $Li(0^+)$.

Now, the initial conditions are inserted.

Due to presence of capacitor C, $V(0^+) = 0$, since the voltage across a capacitor change instantaneously. Also, the current in the inductor L during the time interval -∞ to 0. hence $\phi(0^+) = 0$.

Example

In the circuit of given Figure, the switch S has been open for long time and is closed at t = 0. For e(t)=3u(t), find i(t), t>0.



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Solution

Applying the KVL to the circuit *after S is closed*.

$$e(t) = 3u(t) = 2i(t) + 2\frac{di(t)}{dt} \qquad \text{(i)}$$

Taking the Laplace transform of Eq. (i), we get

$$\frac{3}{s} = 2I(s) + 2\{sI(s) - i(0)\} \qquad \text{(ii)}$$

where i(0) = initial condition.

Before the switch is closed the circuit has reached steady state with inductance acting as a short circuit. Therefore,

$$i(0) = \frac{3}{(1+2)} = 1A$$

From Eq. (ii), we now get

$$3 = 2s(I(s) + 2s^2I(s) - s$$

Rearranging we get

$$I(s) = \frac{s+3}{2s(s+1)} \quad \text{(iii)}$$

Partial fractioning of Eq. (iii) yields the following result.

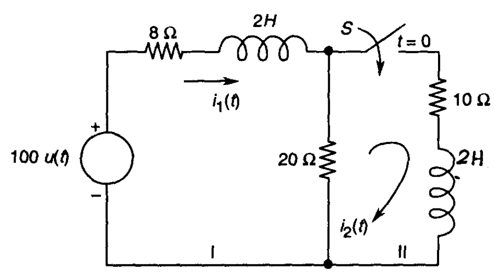
$$I(s) = \frac{1}{2}\left(\frac{3}{s} - \frac{2}{s+1}\right) \quad \text{(iv)}$$

Taking the inverse Laplace transform of Eq. (iv), we get

$$i(t) = (1.5 - e^{-t})u(t)$$

Example

In the circuit of figure the switch S is closed at t = 0. Determine the currents $i_1(t)$ and $i_2(t)$.



Solution

Applying the KVL to loops I and II

$$8i_1(t) + 2\frac{di_1(t)}{dt} + 20\left(i_1(t) - i_2(t)\right) = 100 \quad \text{(i)}$$

$$10i_2(t) + 2\frac{di_2(t)}{dt} + 20\left(i_2(t) - i_1(t)\right) = 0 \quad \text{(ii)}$$

Taking the Laplace transform of Eqs. (i) and (ii), we get

$$8I_1(s) + 2sI_1(s) - 2i_1(0) + 20\left(I_1(s) - I_2(s)\right) = \frac{100}{s} \quad \text{(iii)}$$

$$20\left(I_2(s) - I_1(s)\right) + 10I_1(s) + 2sI_2(s) = 0 \quad \text{(iv)}$$

Before the switch is closed the left part of the circuit has reached steady state within the inductance acting as a short circuit. Therefore,

$$i_1(0) = \frac{100}{28} = 3.57\text{A}$$

Substituting this value in Eq.(iii) and rearranging both eq. (iii) and (iv), we have

$$(2s+28)I_1(s) - 20I_2(s) = \frac{100}{s} + 7.14 \quad \text{(v)}$$

$$-20I_2(s) + (2s+30)I_2(s) = 0 \quad \text{(vi)}$$

From Eq. (vi), we have

$$I_1(s) = \frac{s+15}{10} I_2(s) \quad \text{(vii)}$$

Substituting this in Eq. (v), solving for $I_2(s)$ and factorizing its denominator, we get

$$I_2(s) = \frac{5(100+7.14s)}{s(s+24.5)(s+4.5)} \quad \text{(viii)}$$

By partial fractioning, we can write

$$I_2(s) = \frac{4.54}{s} - \frac{0.77}{s+24.5} - \frac{3.77}{s+4.5} \quad \text{(ix)}$$

Taking the Laplace inverse on both sides

$$i_2(t) = \left(4.54 - 0.77e^{-24.5t} - 3.77e^{-4.5t}\right)u(t) \quad \text{(x)}$$

Now $$I_1(s) = \frac{(s+15)(7.14s+100)}{2s(s+4.5)(s+24.5)} \quad \text{(xi)}$$

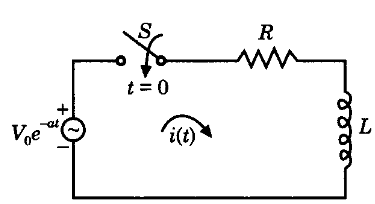
or $$I_1(s) = \frac{6.8}{s} + \frac{0.73}{s+24.5} - \frac{3.96}{s+4.5} \quad \text{(xii)}$$

Taking the inverse Laplace transform, we get

$$i_1(t) = \left(6.8 - 3.96e^{-4.5t} - 0.77e^{-24.5t}\right)u(t) \quad \text{(xiii)}$$

Problem

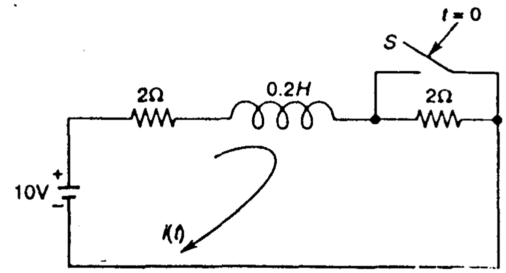
In the series RL circuit shown in figure, determine current i(t).



Answer: $i(t) = \frac{V_0}{L} t e^{-\alpha t}$ *where* $\alpha = R/L$.

Problem

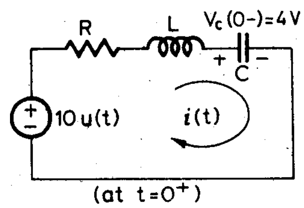
In the circuit of Figure, the switch S has been open for a long time and is closed at t = 0. Find i(t), t>0.



Answer: $i(t) = \left(5 - 2.5e^{-10t}\right)u(t)$

Example

In a series RLC network R = 0.5Ω, L = 1 H and C = 1 F. If the initial voltage on the capacitor is 4 V, find i(t) following switching of a voltage 10 u(t) into the circuit. Assume zero initial condition for the inductor and the polarity of charge on the capacitor as shown in figure.



Solution

Since R = ½ Ω; L = 1H, C = 1F

$$\therefore \qquad Z(s) = R + Ls + \frac{1}{Cs} = \frac{1}{2} + s + \frac{1}{s} = \frac{s^2 + s + 2}{2s}$$

$$\therefore \qquad Y(s) = \frac{2s}{s^2 + s + 2} = \frac{2s}{(s + 0.25 - j0.968)(s + 0.25 + j0.968)}$$

Here Z(s) is impedance and Y(s) is admittance in s domain.

Now $\qquad I(s) = Y(s)V(s)$

or $\qquad i(t) = Y(t)V(t)$

$$= \left[K_1 e^{[-0.25 + j0.968]t} + K_2 e^{[-0.25 - j0.968t}\right] 10u(t)$$

$$= 10Ke^{-0.25t} \cos[0.968t + \phi]u(t)$$

Due to zero initial condition of the inductor

$i(0-) = i(0+) = 0$

$\therefore$ $0 = 10K\cos\phi$

or $\phi = \pm\pi/2$ (giving also K = 0)

Also, $\left.\frac{di}{dt}\right|_{t=0+}$ (i.e. drop across the inductor) = 10 – 4 = 6 V

[Because At t = 0+, drop across the inductor is the difference of supply voltage and initial voltage on the capacitor]

or $6 = 10K[-0.25\cos\phi - 0.968\sin\phi] = 10\,\mathrm{K}\left[-0.25\cos\frac{\pi}{2} - 0.968\sin(-\frac{\pi}{2})\right]$

Solving $K = 0.62$

Hence $i(t) = 0.62\,x10e^{-0.25t}\cos[0.968t + (-\pi/2)]u(t)$

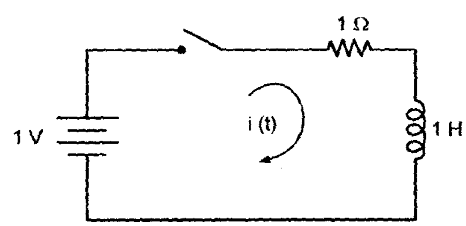
$\therefore$ $i(t) = 6.2e^{-0.25t}\sin(0.968t)u(t)$

Example (AMIE W12, 10 marks)

A series RL circuit is energized by a d.c. voltage of 1.0 V by switching it at t = 0. If R = 1.0 Ω, L = 1.0 H, find the expression of the current using convolution integral.

Solution

See following circuit.



Here $Z(s) = R + sL$ (Assuming zero initial condition)

$\therefore$ $y(s) = \frac{1}{R+sL} = \frac{1}{L}.\frac{1}{s+R/L}$

Taking the inverse transform

$$y(t) = \frac{1}{L}e^{(-R/L)t}$$

Also, in Laplace domain, I(s) = Y(s).V(s) or using convolution integral

$$i(t) = y(t) * v(t)$$

or $i(t) = \int_0^t y(t-\tau)v(\tau)d\tau = \frac{1}{L}\int_0^t e^{(-R/L)(t-\tau)}1d\tau$

$$= \frac{1}{R}(1 - e^{(-R/L)t})$$

However R = 1 Ω and L = 1 H

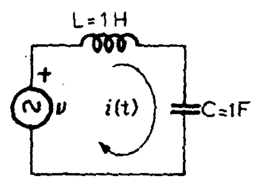
$\therefore \qquad i(t) = 1 - e^{-t}$

Example

In a series LC circuit, the supply voltage being v = V_{max}cos(t), find i(t) at t = 0+ following switching at t = 0 with zero initial conditions. Assume L = 1 H; C = 1 F.

Solution

See following figure.



(Circuit at t=0+ following switching)

Application KVL at t = 0+ in Laplace domain

$$I(s)\left[Ls + \frac{1}{Cs}\right] = Lv + \frac{sV_m}{s^2+1}$$

or
$$I(s)\left[s + \frac{1}{s}\right] = \frac{sV_m}{s^2+1}$$

or
$$I(s) = \frac{sV_m}{(s^2+1)\left(s+\frac{1}{s}\right)} = \frac{V_m s^2}{(s^2+1)(s^2+1)} = \frac{s^2V_m}{(s^2+1)^2}$$

Before, we find the partial fraction expression, the roots are +j, -j, +j and –j.

$$\therefore \qquad I(s) = \frac{s^2V_m}{(s+j)(s-j)(s+j)(s-j)} = \frac{s^2V_m}{(s+j)^2(s-j)^2}$$

$$= \frac{K_1}{(s-j)^2} + \frac{K_1*}{(s+j)^2} + \frac{K_2}{(s-j)} + \frac{K_2*}{s+j}$$

$$\therefore \qquad K_1 = i(s)(s-j)^2\Big|_{s=j} = \frac{s^2V_m}{(s+j)^2}\Bigg|_{s=j} = \frac{j^2V_m}{(2j)^2} = \frac{V_m}{4}$$

$$K_2 = \left|\frac{1}{(2-1)!}\frac{d}{ds}(s-j)^2 I(s)\right|_{s=j} = \left|\frac{(s+j)^2V_m 2s - V_m s^2 x2(s+j)}{(s+j)^4}\right|_{s=j} = -j\frac{V_m}{4}$$

$\therefore \qquad K_1{*} = \frac{V_m}{4}; K_2{*} = j\frac{V_m}{4}$

Thus $\qquad I(s) = \frac{V_m/4}{(s-j)^2} + \frac{V_m/4}{(s+j)^2} + \frac{-jV_m/4}{s-j} + \frac{jBV_m}{s+j}$

$$= \frac{V_m}{4}\left[\frac{1}{(s-j)^2} + \frac{1}{(s+j)^2} - \frac{j}{s-j} + \frac{j}{s+j}\right]$$

Inverse of Laplace transform gives

$$I(t) = \frac{V_m}{4}\left[te^{jt} + te^{-jt} - je^{it} + je^{-jt}\right]$$

$$= \frac{V_m}{4}\left[2t.\frac{e^{jt} + e^{-jt}}{2} - j(2j)\frac{e^{jt} - e^{-jt}}{(2j)}\right]$$

$$= \frac{V_m}{2}[t\cos t + \sin t]$$

ASSIGNMENT

CIRCUIT THEORY AND CONTROL

Q.1. (AMIE W05, S07, 08, 10 marks): State and explain (prove) the following: (i) Initial value theorem (ii) final value theorem (ii) Convolution integral.

Q.2. (AMIE W11, 12 marks): State the initial and final value theorems. Compute the Laplace transform of the function

$$f(t) = (1 + 3e^{-2t} + 4te^{-2t})u(t)$$

Verify the initial value theorem for this function.

Answer: $F(S) = \frac{1}{s} + \frac{3}{s+2} + \frac{4}{(s+2)^2}$; LHS = RHS = 4

Q.3. (AMIE S12, 6 marks): State the time scaling property of Laplace transform. Prove it.

Q.4. (AMIE W06, 8 marks): Evaluate the inverse Laplace transform of $\left[\frac{1}{(s+1)} \cdot \frac{1}{(s+2)}\right]$

Answer: $\frac{e^{-t}}{3}(e^{3t} - 1)$

Q.5. (AMIE W11, 8 marks): Find the inverse transformation of the function

$$F(s) = \frac{s^2 + 6s + 8}{s^3 + 4s^2 + 3s}$$

Answer: $f(t) = \frac{8}{3} - \frac{3}{2}e^{-t} - \frac{1}{6}e^{-3t}$

Q.6. (AMIE S07, 4 marks): Determine the final value of f(t), if

$$F(s) = \frac{s}{(s^2 + 5s + 3)(s+1)}$$

Answer: 0

Q.7. (AMIE S07, 6 marks): Determine

$$L^{-1}\frac{1}{s^2 + 4s + 3}$$

using convolution theorem.

Answer: $\frac{e^{-t}}{4}\left(e^{4t} - 1\right)$

Q.8. (AMIE S08, 8 marks): Find the Laplace transform of the following functions:

(i) $tu(t)$ (unit ramp function)

(ii) $te^{-at}u(t)$

(iii) $\sinh(bt)u(t)$

Answer: (i) $1/s^2$, (ii) $1/(s + a)^2$ (iii) $b/(s^2 - b^2)$

Q.9. (AMIE S08, 7 marks): Obtain the inverse Laplace transform of the following:

$$\frac{12(s+2)}{s(s^2+4s+8)}$$

Answer: $3\left[1+3e^{-2t}\cos 2t\right]$

Q.10. (AMIE S13, 5 marks): Current I(S) in a network is given by

$$I(\mathrm{S}) = \frac{2S+3}{S^2+3S}$$

Find i(t) the current at any time "t".

Answer: $i(t) = u(t) + e^{-3t}$

Q.11. (AMIE W08, 7 marks): Find the inverse Laplace transform of the following:

$$X(s) = \frac{1}{(s^2+5^2)^2}$$

Answer: $f(t) = \frac{1}{250}\left[\sin 5t - 5t\cos 5t\right]$

Q.12. (AMIE S10, 5 marks): Find the inverse Laplace transform for

$$F(s) = (7s+2)/(s^3+3s^2+2s).$$

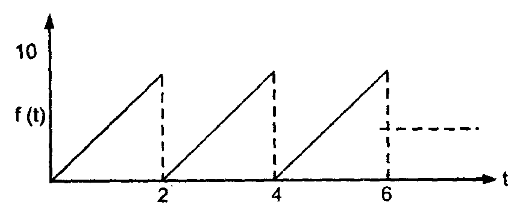
Answer: $f(t) = u(t) + 5e^{-t} - 6e^{-2t}$

Q.13. (AMIE S09, 2 marks): Derive the Laplace transform of the function f(t) = t.

Answer: $1/s^2$

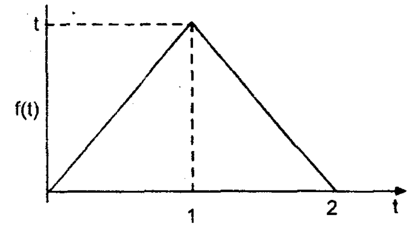
Q.14. (AMIE W11, 5 marks): Only one half cycle (starting t = 0) is present for a sinusoidal wave of amplitude 2V and time period 0.02 s. Find the time domain equation and calculate the Laplace transform for this half cycle.

Q.15. (AMIE W09, 8 marks): Determine the Laplace transform of the following periodic function.



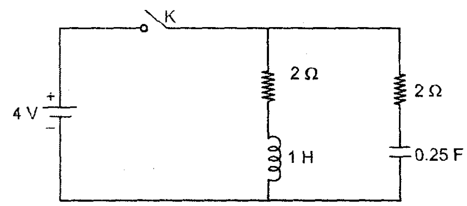
Answer: $\frac{5}{s} - \frac{10}{\pi}\left[\frac{\omega}{s^2+\omega^2} + \frac{1}{2}.\frac{2\omega}{(s^2+4\omega^2)} + \frac{1}{3}.\frac{3\omega}{(s^2+9\omega^2)} + ...\right]$

Q.16. (AMIE S10, 7 marks): For the given signal, find the Laplace transform.



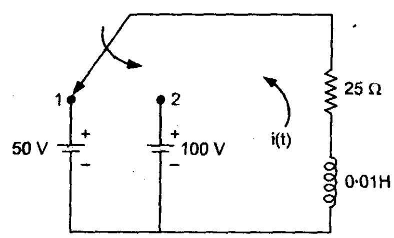
Answer: $f(t) = u(t) + 5e^{-t} - 6e^{-2t}$

Q.17. (AMIE S09, 6 marks): In the given network, switch K is opened at time t = 0, the steady state having established previously. With switch K open, draw the transform (s-domain) network representing all elements and all initial conditions. Write the transform equation for current in the loop. From that expression, also find the current i(t) in the loop.



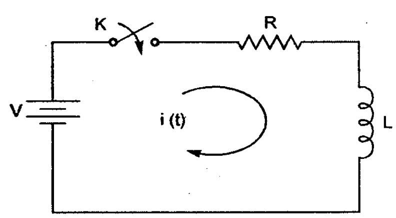
Answer: $2e^{-2t}(1-2t)$

Q.18. (AMIE S05, 6 marks): In the network shown in given figure, the switch is kept in position 1 for a long time and then moved to position 2 at t = 0. Determine the current expression i(t) using Laplace transform.



Answer: $i(t) = 4 - 2e^{-2500t}$

Q.19. (AMIE S06, 8 marks): Find the particular solution of the circuit shown below.

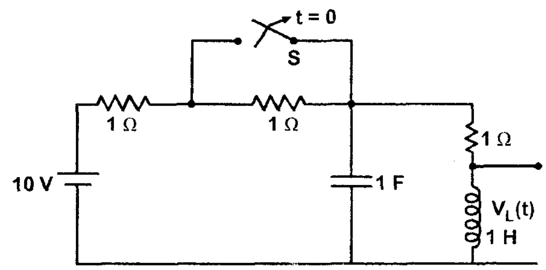


Answer: $i(t) = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)$

Q.20. (AMIE S12, 8 marks): A step voltage V(t) = 100 u(t) is applied to a series RLC circuit with L = 10 H, R = 2 Ω and C = 5F. The initial current in the circuit is zero but there is an initial voltage of 50 V on the capacitance in a direction which opposes the applied source. Find the expression for the current in the circuit.

Answer: $i(t) = (0.5 - j0.33)e^{(0.2+j0.3)t} + (0.5 + j0.33)e^{-(0.2+j0.3)t}$

Q.21. (AMIE W12, 10 marks): The circuit shown in figure is initially under steady state condition with switch S closed. Switch S is opened at t = 0. Find the voltage across the inductance, L, as function of "t". Use Laplace transform method.



CIRCUIT AND FIELD THEORY

Q.22. (AMIE W07, 08, S05, 09, 10 marks): Write short notes on Initial and Final value theorems.

Q.23. (AMIE S08, 12, 10 marks): State and prove (i) convolution theorem (ii) complex translation theorem of Laplace transform.

Q.24. (AMIE S10, 6 marks): What is convolution? State and prove convolution theorem.

Q.25. (AMIE W11, 10 marks): Establish analytically the concept of convolution using Laplace transformation of two functions.

Q.26. (AMIE W08, 3 marks): A function in Laplace domain is given by

$$F(s) = \frac{2}{s} - \frac{1}{s+3}$$

Obtain its value by final value theorem in time domain.

Answer: 2

Q.27. (AMIE S05, 6 marks): Find the value of i(0) using the initial value theorem for the Laplace transform given below:

$$I(s) = \frac{2s+3}{(s+1)(s+3)}$$

Obtain its inverse form.

Answer: 2

Q.28. (AMIE W12, 10 marks): State and briefly explain the initial and final value theorem in Laplace domain. A function in Laplace domain is given by

$$F(s) = \frac{2(s+4)}{(s+3)(s+8)}$$

Find the initial and final values.

Answer: 1, 0

Q.29. (AMIE W08, 5 marks): If $f_1(t) = 2u(t)$ and $f_2(t) = e^{-3t}u(t)$, determine the convolution between $f_1(t)$ and $f_2(t)$.

Q.30. (AMIE W08, 7 marks): A 10 V step voltage is applied across a RC series circuit at t = 0. Find i(t) at t = 0+ and obtain the value of $(di/dt)_{t=0}$ assuming R = 100 Ω and C = 100 μF.

Answer: 0

Q.31. (AMIE W09, 8 marks): In a Laplace domain, a function is given by

$$F(S) = M\left[\frac{(s+\alpha)\sin\theta}{(s+\alpha)^2+\beta^2} + \frac{\beta\cos\theta}{(s+\alpha)^2+\beta^2}\right]$$

Show by initial value theorem

$$\lim_{t\to 0} f(t) = M\sin\theta$$

Q.32. (AMIE S05, 4 marks): A pulse voltage of width 2 seconds and magnitude 10 volts is applied at time t = 0 to a series R-L circuit consisting of resistance R = 4Ω and inductor L = 2 Henry. Find the current i(t) by using Laplace transformation method. Assume zero current through the inductor L before application of the voltage pulse.

Answer: $\left(\frac{5}{2} - e^{-5t}\right)u(t) - 5\left(1-e^{-(t-2)}\right)^{u(t-2)}$

Q.33. (AMIE S10, 8 marks): A step voltage of 100 t u(t) volts is applied across a series RC circuit where R = 5 K-ohm and C = 4 μF. Find the voltage drop across the resistor R and show that it is approximately equal to 2V.

Q.34. (AMIE S09, 5 marks): A function in Laplace domain is given by

$$I(s) = \frac{s+1}{s(s^2+4s+4)}$$

Obtain its inverse transform.

Answer: $i(t) = \frac{1}{4}u(t) + \frac{1}{2}te^{-2t} + \frac{1}{2}e^{-2t}$

Q.35. (AMIE W09, S10, 7 marks): Obtain inverse Laplace transform of I(s) when

$$I(s) = \frac{250}{(s^2+625)(s+2)}$$

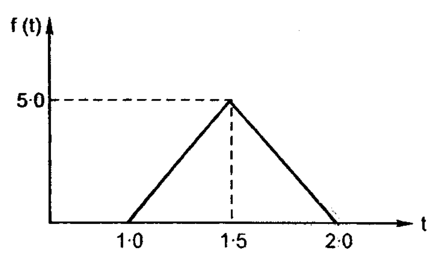
Answer: $i(t) = 0.4e^{-2t} - \frac{5}{(25+j2)}e^{-j25t} - \frac{5}{(25-j2)}e^{j25t}$

Q.36. (AMIE S08, 12, 12 marks): Find the inverse Laplace transform of the following

(i) $F(s) = \frac{s+1}{s^3+s^2-6s}$ (ii) $F(s) = \frac{s+2}{s^5-2s^4+s^3}$

(i) Answer: $f(t) = -\frac{1}{6} + \frac{3}{10}e^{2t} - \frac{2}{15}e^{-3t}$ (ii) $f(t) = 8+5t+t^2-8e^t+te^t$

Q.37. (AMIE S09, 5 marks): A pulse waveform is shown in following figure. Obtain its Laplace transform.



Answer: 20

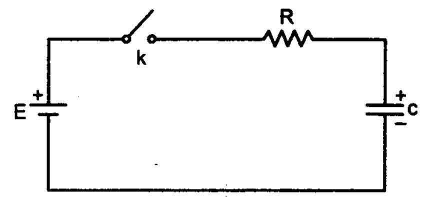
Q.38. (AMIE S05, 8 marks): Find the current i(t) in a series RC circuit consisting R = 2Ω and C = 1/4 farad when each of the following driving force voltage is applied:

(i) ramp voltage 2r(t - 3)

(ii) step voltage 2u(t - 3)

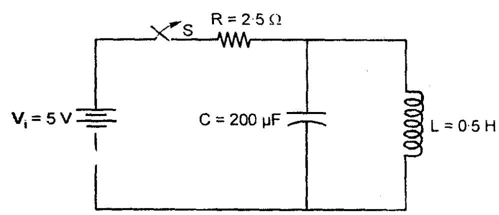
Answer: (i) $i(t) = \frac{1}{2}\left[u(t-3) - e^{-2(t-3)}.u(t-3)\right]$ (ii) $i(t) = e^{-2(t-3)}.u(t-3)$

Q.39. (AMIE W05, 8 marks): Find the current response i(t) when a step voltage is applied by closing the switch k. Assume Q_0 be the initial charge on the capacitor. Use Laplace transform method.



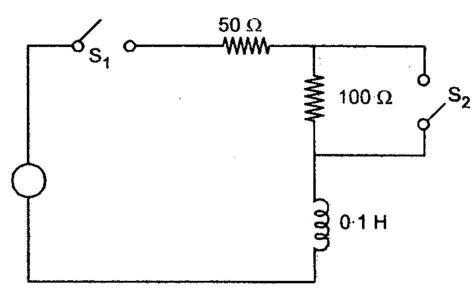
Answer: $i(t) = \left(\frac{E}{R} - \frac{Q_0}{Rc}\right)e^{-t/Rc}$

Q.40. (AMIE S05, 8 marks): In the network, the switch S is closed and a steady state is attained. At t = 0, the switch is opened. Determine the current through the inductor.



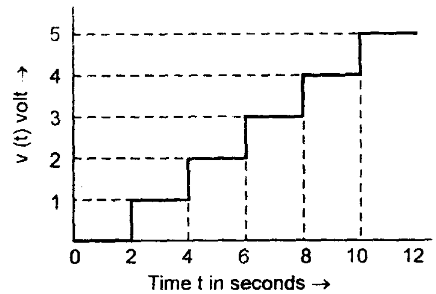
Answer: $i(t) = 4\cos 10^4 t$

Q.41. (AMIE S09, 5 marks): In following figure, switch S is closed at t = 0. Switch S_2 is opened at t = 4 ms. Obtain I for t > 0.



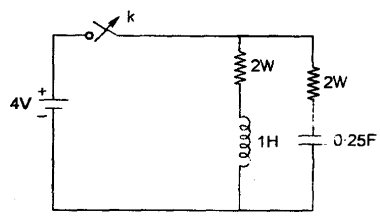
Q.42. (AMIE W06, 8 marks): Prove that the Laplace transform of any time function f(t) delayed by time a is e^{-as} times the transform of the function F(s).

Q.43. (AMIE W06, 8 marks): A staircase voltage v(t) shown in figure is applied to an RL network consisting of L = 1 H and R = 2Ω. Write the equation for the staircase voltage in terms of step function. Find the Laplace transform of v(t). Find the current i(t) in the circuit. Draw the waveform of current i(t). Assume zero current through the inductor L before applying the voltage.



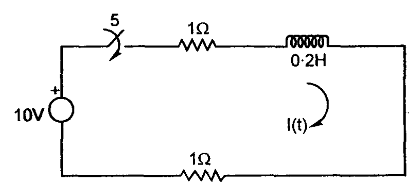
Answer: $u(t-2)+u(t-4)+u(t-6)+u(t-8)+u(t-10)-5u(t-12)$

Q.44. (AMIE W06, 10, 10 marks): In the given network, switch K is opened at time t = 0, the steady state having established previously. With switch K open, draw the transform network representing all elements and all initial conditions. Write the transform equation for current in the loop. Also find the current i(t) in the loop.



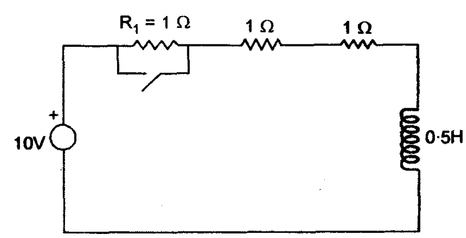
Answer: $i(t) = -4te^{-2t} + 2e^{-2t}$

Q.45. (AMIE W07, 8 marks): In following figure, obtain the expression of transient current using Laplace transform, when the switch is suddenly closed at time t = 0. Also plot i(t) vs. t.



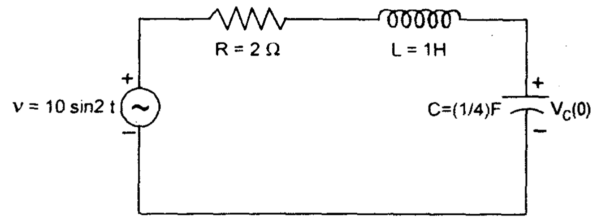
Answer: $i(t) = 5(1-e^{-5t})$

Q.46. (AMIE W07, 12 marks): In given figure, the circuit is connected to voltage source t = 0+. After 0.1 sec, resistance R_1 is suddenly short circuited. Using Laplace transform, obtain the expression of current for time t = 0+ to t = 0.1 sec and t = 0.1 sec to t = ∞ sec.



Answer: $i(t) = 5.6833 - \frac{10}{3}e^{-0.25t}$

Q.47. (AMIE S08, 12, 8 marks): For the figure shown, find the current i(t) using Laplace transform method. Given that i(0+) = 2 A and $v_c(0+) = 4$ V.

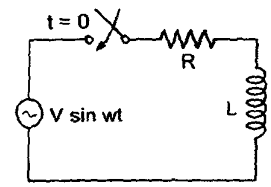


Answer: $I(s) = \frac{2(s^2 - 10s + 8)}{(s^2 + 4)(s^2 + 2s + 4)}$; Now find inverse Laplace transform using partial fraction method.

Q.48. (AMIE S12, 10 marks): Find an expression for the value of current at any instant after a sinusoidal voltage of amplitude 600 V at 50 Hz applied to a series circuit of resistance 10 ohm and inductance 0.1 Henry, assuming that the voltage is zero at the instant of switching (t = 0). Also, find the value of transient current at t = 0.02 sec.

Answer: 2.34 Amp.

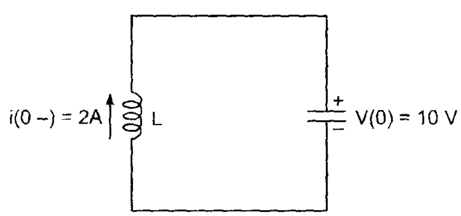
Hint:



Q.49. (AMIE S08, 10 marks): Find an expression for the value of current at any instant after a sinusoidal voltage of amplitude 600 V at 50 Hz is applied to series circuit of resistance 10 Ω and inductance 0.1 H, assuming that the voltage is zero at the instant of switching (t = 0). Also, find the value of transient current at t = 0.02 sec.

Answer: $-97.42 - j[5.915 \cos 0.066]$

Q.50. (AMIE W11, 10 marks): In a LC circuit shown in figure, the initial current through the inductor being 2 A, the initial voltage is 10 V. Assume L = 1 H and C = 0.5 F. Find the voltage across the capacitor at t = (0+) using Laplace transformation technique.



Answer: $V(t) = 10\left[\cos\sqrt{2}t + \frac{1}{\sqrt{2}}\sin\sqrt{2}t\right]$